

1 Generations Rules

Theorem 1. *For a generations rule with C states, the minimum possible oscillator period is p .*

Proof. For an oscillator to oscillate at a period p , there must be a cell that oscillates at that period p . However, any cell that dies will have to go through $C - 1$ states where it is dead before it can be born again. Hence, no oscillator can have a period $p < C$. $\square$

For the subsequent proofs, we will need to make sure of a representation of the ship as a doubly-infinite sequence of rows, first introduced by David Eppstein in [1].

Definition 1. *Let r is a sequence of rows representing the ship with its phases interleaven (this is the gfind-representation of the ship). This infinite sequence of rows must obey $r_{i-p+k} = \text{evolve}(r_i, r_{i-p}, r_{i-2p})$ in order for this to be a valid spaceship. We will define the quantity $p-k$ as δ and the quantity $p-nk$ as δ_n . For cases of $\gcd(k, p) > 1$, the rows will obey $r_{i-p+k \bmod p} = \text{evolve}(r_i, r_{i-p}, r_{i-2p})$ where*

$$k_i = \begin{cases} k & \text{if } i < p - k \\ k + 1 & \text{if } i < p - 1 \\ 1 & \text{else} \end{cases}$$

Theorem 2. *For an range-1 Moore isotropic non-totalistic generations rule with C states and no survival conditions, no ship with period $1 < p \leq C$ and $k = 1$ can exist.*

Proof. Let r_i denote the first empty row in the sequence where a cell is born. The cell that will be born is denoted as r_i^x . Now, we know that there are no survival conditions, hence, r_{i+k}^x will immediately die and $r_{i+nk}^x = n \bmod C$ where $n \leq C$.

If $k = 1$, then $r_{i+p}^x = r_{i+pk}^x = p \bmod C$, which will be a dying state unless $p = 1$.

This implies that r_{i+p}^{x+1} and r_{i+p}^{x-1} must both be alive, since having the B1c birth transition will completely prevent the existence of ships.

Without loss of generality, we will now focus on r_{i+p}^{x+1} . Its predecessor is $r_{i+p-k}^{x+1} = r_{i+\delta}^{x+1}$ where $\delta = p-1$. The possible neighbours are $\{r_{i+\delta}^x, r_{i+\delta+p}^x, r_{i+\delta+p}^{x+1}, r_{i+\delta+p}^{x+2}\}$.

- $r_{i+\delta}^x = \delta \bmod C$. If $p = 2$, then cell is live. If not, it is dead.
- $r_{i+\delta+p}^x$ is dead if $2p - 1 \leq C$. If not, its state is can be either dead or alive.
- $r_{i+\delta+p}^{x+1}$ is dead, since $p \leq C$.
- $r_{i+\delta+p}^{x+2}$ can either dead or alive.

Case 1: If $p = 2$, then $r_{i+\delta}^x = 1$. Then, $2p - 1 \leq C$ and $r_{i+\delta+p}^x$ is also dead. Hence, the only other cell that can be alive is $r_{i+\delta+p}^{x+2}$. Now, this cell will have predecessor $r_{i+2\delta}^{x+2}$. Then, the exact same argument will apply on this cell, meaning no finite $c/2o$ front can be formed.

Case 2: If $p > 2$, then $r_{i+\delta}$ is dead. However, then $r_{i+\delta+p}^x$ may be alive if $2p - 1 > C$. If that is not the case, the ship cannot exist. Now, assuming $2p - 1 > C$, the neighbours that we will then be concerned about it $r_{i+\delta+p}^x$ and $r_{i+\delta+p}^{x+2}$. Without loss of generality, we can once again, focus on $r_{i+\delta+p}^{x+2}$ and its predecessor, $r_{i+2\delta}^{x+2}$. Then, we can apply the same argument on this cell, meaning no finite front can be formed.

Hence, no ships with period $1 < p \leq C$ and $k = 1$ can exist. $\square$

Remark 1. This explains the strange lack of $c/2o$ and $c/3o$ ships in Brian's Brain ($/2/3$).

Conjecture 1. For an range-1 Moore isotropic non-totalistic generations rule with C states and no survival conditions, no ship with period $1 < p \leq C$ can exist.

Lemma 1. *Let r_i be the last row in a spaceship to have a live cell. Then, all live cells in $r_i - k + 1$ to r_i must die and no cells must to born and r_{i+p} will be ... (please see proof).*

Proof. We will need to look at r_{i+p} 's first predecessor to have live cells, $r_{i-k+\delta_n}$ where $n = \lceil \frac{p}{k} \rceil$. In this row, all live cells must die and none must be born, if not rows below r_i will have live cells. This applies to all rows between r_i (inclusive) and r_{i-k} (exclusive).

Then, r_{i+p} will be

$$r_{i+p}^x = \begin{cases} (r_{i+p-nk}^x + n) \bmod C & \text{if } (r_{i+p-nk}^x + n) \leq C, r_{i+p-nk}^x \neq 0 \\ 0 & \text{else} \end{cases}$$

In the special case of $k = 1$, $r_{i+p-nk} = r_i$ and $n = p$. $\square$

Remark 2. We can look at the neighbour to this row, $r_{i-k+\delta_n+p}$. The first predecessor of this row with live cells is $r_{i-k+2\delta_n}$.

We can continue this process $k - 1$ times and the predecessor rows will be $r_{i-k+(m\delta_n \bmod k)}$, with $m < k$.

The neighbouring row of the final predecessor row will be $r_{i+p-k+((k-1)\delta_n \bmod k)}$.

$$\begin{aligned} & i + p - k + ((k-1)\delta_n \bmod k) \\ &= i + p - k + (((k-1)(p \bmod k) \bmod k) \\ &= i + p - k + (k(p \bmod k) - (p \bmod k)) \bmod k \\ &= i + p - k - (p \bmod k) \\ &= i + (n-1)k \end{aligned}$$

Note that this forms a cycle, which may be useful for proving statements about the trailing edge when $k > 1$.

Theorem 3. *For a range-1 Moore outer-totalistic generations rules with B23 and S0, the trailing edge of a ship with period p and displacement $k = 1$ cannot die if $1 < p < C$.*

Proof. We cite the proof for 2-state range-1 Moore outer-totalistic rules by David Eppstein that for rules with B23 and S0, the trailing edge of any pattern cannot die. For a generations rule, this is not true, as dying cells below the live trailing edge can prevent cells from being born.

TODO: Complete the proof.

Remark 3. This shows that 0/23/3 cannot have a $c/2o$ while 0/23/4, 0/23/5, ..., can have a $c/2$. This is consist with empirical findings.

References

1. David Eppstein. Searching for spaceships. *CoRR*, cs.AI/0004003, 2000.